# SOME RESULTS ON MULTIPLCITY-FREE SPACES

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Abstract. Let $(M,\omega)$ be a connected symplectic manifold on which a connected Lie group $G$ acts properly and in a Hamiltonian fashion with moment map $\mu : M \longrightarrow \mathfrak{g}^*$. Our purpose is investigate multiplicity-free actions, giving criteria to decide a multiplicity freenes of the action. As an application we give the complete classification of multiplicity-free actions of compact Lie groups acting isometrically and in a Hamiltonian fashion on Hermitian symmetric spaces of noncompact type. Successively we make a connection between multiplicity-free actions on $M$ and multiplicity-free actions on the symplectic reduction and on the symplectic cut, which allow us to give new examples of multiplicity-free actions.

## 1. Introduction

Let $(M,\omega)$ be a connected symplectic manifold with a proper and Hamiltonian action of a connected Lie group $G$ and let $\mu : M \longrightarrow \mathfrak{g}^*$ be a corresponding moment map. In 1984 Guillemin and Sternberg [10], motivated by geometric quantization, introduced the notion of multiplicity-free space when the ring of the $G-$invariant functions on $M$ is commutative with respect to the Poisson-bracket. The manifold $M$ is called $G$ multiplicity-free space and the $G-$action is called multiplicity-free. The term multiplicity-free comes from the representation theory of Lie groups.

A unitary representation of a Lie group $G$ on a Hilbert space $\mathbb{H}$ is said to be multiplicity-free if each irreducible representation of $G$ occurs with multiplicity zero or one in $\mathbb{H}$. The relationships between the two definitions comes via the theory of geometric quantization. The condition that a unitary representation of $G$ on $\mathbb{H}$ be multiplicity-free is equivalent to the condition that the ring of bounded operators on $\mathbb{H}$ be commutative.

In this paper we investigate multiplicity-free actions, which we also may call coisotropic actions, on a symplectic manifold $M$, imposing that $G$ acts properly and in a Hamiltonian fashion on $M$ and a technical condition, which is needed for applying the symplectic slice, see [2] and [23], and the symplectic stratification of the reduced space given in [2], [23], which is the following.

For every $\alpha \in \mathfrak{g}^*$, $\mathfrak{g}^*$ is the dual of the Lie algebra of $G$, $G\alpha$ is a locally closed coadjoint orbit of $G$. Observe that the condition of a coadjoint orbit being locally closed is automatic for reductive groups and for their semidirect products with vector spaces. There exists an example of a solvable group due to Mautner [28] p.512, with non-locally closed coadjoint orbits.

One of our purpose is to give Equivalence theorem for multiplicity-free action, which shall allow us to prove that the complete classification of compact Lie groups acting multiplicity-free on irreducible Hermitian symmetric spaces of noncompact type follows from one given in a compact case.

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We also prove a reduction principle for multiplicity-free actions and we make a connection between multiplicity-free actions on $M$ and multiplicity-free actions on the symplectic reduction and on the symplectic cut, mainly in a Kähler geometry. As an application we give new examples of multiplicity-free actions on compact Kähler manifolds which are not Hermitian symmetric spaces.

## 2. Hamiltonian viewpoint

Let $M$ be a connected differential manifold equipped with a non-degenerate closed $2-$form $\omega$. The pair $(M, \omega)$ is called symplectic manifold. Here we consider a finite-dimensional connected Lie group acting smoothly and properly on $M$ so that $g^*\omega = \omega$ for all $g \in G$, i.e. $G$ acts as a group of canonical or symplectic diffeomorphism. For $f, g \in C^\infty(M)$, define $\{f, g\} = \omega(X_f, X_g)$, where $X_f$ and $X_g$ are the vector fields which is uniquely defined by $df = i_{X_f}\omega$ and $dg = i_{X_g}\omega$. It follows that $(C^\infty(M), \{,\})$ is a Poisson algebra.

The $G$-action is called *Hamiltonian*, and we said that $G$ acts in a Hamiltonian fashion, if there exists a map

$$\mu : M \longrightarrow \mathfrak{g}^*,$$

which is called moment map, satisfying:

(1) for each $X \in \mathfrak{g}$ let
- $\mu^X : M \longrightarrow \mathbb{R}$, $\mu^X(p) = \mu(p)(X)$, the component of $\mu$ along $X$,
- $X^\#$ be the vector field on $M$ generated by the one parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\} \subseteq G$.

Then

$$\mathrm{d}\mu^X = \mathrm{i}_{X^\#}\omega$$

i.e. $\mu^X$ is a Hamiltonian function for the vector field $X^\#$.

(2) $\mu$ is $G-$equivariant, i.e. $\mu(gp) = Ad^*(g)(\mu(p))$, where $Ad^*$ is the coadjoint representation on $\mathfrak{g}^*$.

Let $x \in M$ and $\mathrm{d}\mu_x : T_xM \longrightarrow T_{\mu(x)}\mathfrak{g}^*$ be the differential of $\mu$ at $x$. Then

$$Ker\mathrm{d}\mu_x = (T_xG(x))^{\perp_\omega} := \{v \in T_xM : \omega(v, w) = 0,\ \forall w \in T_xG(x)\}. \tag{1}$$

If we restrict $\mu$ to a $G-$orbit $Gx$, then we have the homogeneous fibration $\mu : Gx \longrightarrow Ad^*(G)\mu(x)$ and the restriction of the ambient symplectic form $\omega$ on the orbit $Gx$ is the pullback by the moment map $\mu$ of the symplectic form on the coadjoint orbit through $\mu(x)$

$$\omega_{|Gx} = \mu^*(\omega_{Ad^*(G)\mu(x)})_{|Gx} \tag{2}$$

see [2] p. 211, where $\omega_{G\mu(x)}$ is the Kirillov-Konstant-Souriau (KKS) symplectic form on the coadjoint orbit of $\mu(x)$ in $\mathfrak{g}^*$. This implies the following result.

**Proposition 2.1.** *A $G$-orbit $Gx$ is a symplectic submanifold of $M$ if and only if the moment map restricted to $Gx$ into $G\mu(x)$, $\mu_{|Gx} : Gx \longrightarrow G\mu(x)$, is a covering map. In particular if $G$ is a compact Lie group then $G_x = G_{\mu(x)}$ and $\mu_{|Gx} : Gx \longrightarrow G\mu(x)$ is a diffeomorphism onto.*

*Proof.* The first affirmation follows immediately from (2). If $G$ is compact, coadjoint orbits are of the form $G/C(T)$, where $C(T)$ is the centralizer of the torus $T$. In particular such orbits are simply connected, form which one may deduce $G_x = G_{\mu(x)}$. □

## 3. Multiplicity-free spaces

Let $(M,\omega)$ be a connected symplectic manifold and let $G$ be a connected Lie group acting properly and as a group of symplectic diffeomorphism on $M$.

**Definition 3.1.** The $G$-action is called *multiplicity-free*, $M$ is called a $G$ multiplicity-free space, if the space of invariant function $C^\infty(M)^G$ is a commutative Lie algebra with respect the Poisson bracket.

By the definition follows that if $K \subset G$ and $M$ is a $K$ multiplicity-free space then $M$ is a $G$ multiplicity-free space as well.

The multiplicity-free actions are also called *coisotropic actions*. This is justified by the following discussion.

**Definition 3.2.** A submanifold $N$ of a symplectic manifold $(M,\omega)$ is said to be *coisotropic* if and only if for every $x \in N$, $(T_xN)^{\perp_\omega} \subset T_xN$. In particular a $G$-action on $M$ is called coisotropic if and only if there exists an open dense subset $U \subset M$ with $Gx$ coisotropic for every $x \in U$.

**Lemma 3.3.** *$M$ is a $G$ multiplicity-free space if and only the $G$-action on $M$ is coisotropic.*

*Proof.* First of all we note the following easy fact: if $f \in C^\infty(M)^G$ then for every $\xi \in \mathfrak{g}$ we have $\{f, f_\xi\} = 0$, where $f_\xi$ is defined by $f_\xi(x) = \mu(x)(\xi)$.

Assume that a generic orbit $Gx$ is coisotropic.

Let $f, g \in C^\infty(M)^G$. Since $\{f, f_\xi\} = \{g, f_\xi\} = 0$ we have $X_f, X_g \in (T_xGx)^{\perp_\omega} \subset T_xGx$, since $Gx$ is coisotropic, for every $x \in U$. Hence

$$\{f,g\}(x) = \omega(X_f, X_g) = 0, \ \forall x \in U,$$

which implies $\{f,g\} = 0$, since $U$ is an open dense subset.

Vice-versa, let $x \in M$ be a regular point. By the slice-theorem there are functions $f_1, \ldots, f_k \in C^\infty(M)^G$ with $df_1 \wedge \ldots \wedge df_k \neq 0$ in some neighborhood $W$ of $Gx$ and

$$Gx = \{y \in W : f_1(y) = \ldots = f_k(y) = 0\}.$$

From $\{f_i, f_j\} = 0$ one may deduce that $X_{f_i} \in T_xGx$. Therefore, since $X_{f_i} \in (T_xGx)^{\perp_\omega}$, $i = 1, \ldots, k$ and $X_{f_1}, \ldots, X_{f_k}$ are independent in $W$, we have that $Gx$ is a coisotropic submanifold. $\square$

*Remark* 3.4. Our proof is essentially one given in [13]. However in [13] the authors assumed that $G$ is a compact Lie group, their proof works also when the $G$-action is a proper action.

It is standard that given a $G$-orbit $Gx = G/G_x$, study the slice representation, i.e. the linear representation of $G_x$ induced from the $G$-action on $T_xM/T_xGx$. In [13] p. 274, as a consequence of the arguments used in the Restriction Lemma, it was proved that given a complex orbit $Gp = G/G_p$ then $G$ acts coisotropically on $M$ if and only if $G_p$ acts coisotropically on the slice, whenever $M$ is a compact Kähler manifold and $G$ is a subgroup of its full isometric group. Here, we give the same result in symplectic context.

**Proposition 3.5.** *Let $(M,\omega)$ be a symplectic manifold and let $G$ be a Lie group which acts properly and in a Hamiltonian fashion on $M$ with moment map $\mu : M \longrightarrow \mathfrak{g}^*$. Let $Gx$ be a symplectic orbit. If $M$ is a $G$ multiplicity-free space then the slice representation is a multiplicity-free representation. Moreover, if $G$ is compact the vice-versa holds as well.*

*Proof.* It follows from symplectic slice, see [2], [19] [23].

There exists a neighborhood of the orbit $Gx$ which is $G$-equivariantly symplectomorphic to a neighborhood of the zero section of the symplectic manifold $(Y = G \times_{G_x} ((\mathfrak{g}_\beta/\mathfrak{g}_x)^* \oplus V), \tau)$, see [2], [23] for an explicit description of the symplectic form $\tau$, with a $G$-moment map $\mu$ given by

$$\mu([g, m, v]) = Ad(g)(\beta + j(m) + i(\mu_V(v))),$$

where $\beta = \mu(x)$, $i : \mathfrak{g}_x^* \hookrightarrow \mathfrak{g}^*$ is the transpose of the projection $p : \mathfrak{g} \longrightarrow \mathfrak{g}_x$, $j : (\mathfrak{g}_\beta/\mathfrak{g}_x)^* \hookrightarrow \mathfrak{g}_x^o$ ($\mathfrak{g}_x^o$ is the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$) is defined by a choice of a $G_x$-equivariant splitting and finally $\mu_V$ is the moment map of the $G_x$-action on the symplectic subspace $V$ of $(T_xGx, \omega(x))$. Note that $V$ is isomorphic to the quotient $((T_xGx)^{\perp_\omega}/((T_xGx)^{\perp_\omega} \cap T_xGx))$. In the sequel we denote by $\omega_V = \omega(x)_{|V}$.

Since $Gx$ is symplectic, $(Y = G \times_{G_x} V, \tau)$ and the moment map $\mu$ becomes

$$\mu([g, m]) = Ad^*(g)(\beta + i(\mu_V(m))),$$

Assume that $M$ is a $G$-multiplicity-free space and let $y = [e, m] \in Y$ be such that $Gy$ is coisotropic.

Let $Y \in Ker\mathrm{d}(\mu_V)_m$. Noting $\mathrm{d}\mu_{[e,m]}(0, Y) = \mathrm{d}(\mu_V)_m(Y) = 0$, which implies, from (1), $Y \in (T_yGy)^{\perp_\omega} \subset T_yGy$. This claims $Y \in T_yGy \cap V = T_mG_xm$, i.e the slice representation is multiplicity-free.

Assume that $G$ is a compact Lie group. It is well known

$$\mathrm{cohom(G, M) = cohom(G_x, V)},$$

which follows from the classical slice theorem for proper actions [20], and rk($G$)=rk($G_x$), since $Gx$ is a symplectic manifold, where cohom($G$,$M$) denotes the codimension of a principal orbit and for every compact group $K$, rk($K$) denotes the rank, namely the dimension of the maximal torus. If $G_x$ acts multiplicity-free on $V$ then cohom($G_x, V$)=rk($G_x$)-rk($G_{\mathrm{princ}}$), see [13], where $G_{\mathrm{princ}}$ is the principal isotropy subgroup of the action, which implies that

$$\mathrm{cohom}(G, M) = rk(G) - rk(G_{\mathrm{princ}}).$$

Therefore, from Theorem 3 [13] p. 269, we get $G$ acts multiplicity-free on $M$. □

**Corollary 3.6.** *Let $M$ be an irreducible Hermitian irreducible symmetric space of non compact type. Let $G$ be a compact group which acts in a Hamiltonian fashion on $M$. Then $G$ acts multiplicity-free on $M$ if and only if it acts multiplicity-free on $M^*$, the corresponding irreducible Hermitian symmetric space of compact type.*

*Proof.* Since $G$ is compact it has a fixed point $x \in M$, from a Theorem of Cartan, see [12]. Hence $G$ acts multiplicity-free on $M$ if and only if $G$ acts multiplicity-free on the tangent space at $x$ if and only if it acts multiplicity free on $M^*$. □

*Remark* 3.7. Corollary 3.6 classifies completely the compact Lie groups acting in a Hamiltonian fashion and coisotropically on the irreducible Hermitian symmetric spaces of noncompact type, due the results proved in [5], [6], [21].

## 4. Equivalence Theorems for multiplicity-free action

From now on we assume that $(M, \omega)$ is a connected symplectic manifold acted on properly and in a Hamiltonian fashion by a connected Lie group $G$. We denote by $\mu : M \longrightarrow \mathfrak{g}^*$ the corresponding moment map for the $G$-action on $M$.

Let $\alpha \in \mathfrak{g}^*$. We define the corresponding reduced space

$$M_\alpha = \mu^{-1}(G\alpha)/G,$$

to be the topological quotient of the subset $\mu^{-1}(G\alpha)$ of $M$ by the action of $G$. It is well known, see [1], [2], [23], that the reduced space $M_\alpha$ is a union of a symplectic manifolds and it can be endowed with a Poisson structure which arise from Poisson structure on the orbits space. Hence $M_\alpha$ is a symplectic stratified space and the manifolds which decompose $M_\alpha$ are called pieces.

Here we analyze the case when $G = G_1 \times G_2$, where $G_i,\ i = 1,2$ are closed connected subgroup of $G$. We assume also that the coadjoint orbits of $G, G_1$ and $G_2$ are locally closed spaces.

Obviously $\mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$ and the moment map $\mu = \mu_1 + \mu_2$, where $\mu_i$ is the corresponding moment map for the $G_i$-action on $M$, $i = 1,2$. Since $\mu$ is $G$-equivariant, we have that $\mu_1$ is invariant under $G_2$ and $\mu_2$ is invariant under $G_1$.

Let $\alpha = \alpha_1 + \alpha_2$. The $G_1$-action on the pieces of the reduced space $M_{\alpha_2} = \mu_2^{-1}(G_2\alpha_2)/G_2$ is symplectic. These moment maps on the pieces fit together to form an application

$$\mu_{1,2} : M_{\alpha_2} \longrightarrow \mathfrak{g}_1^*,$$

such that $\mu_{1,2} = \mu_1 \circ \pi_2$, where $\pi_2$ is the projection on $M_{\alpha_2}$. Clearly $G_2$ acts on the reduced space $M_{\alpha_1} = \mu_1^{-1}(G_1\alpha_1)/G_1$ with moment map

$$\mu_{2,1} : M_{\alpha_1} \longrightarrow \mathfrak{g}_2^*$$

such that $\mu_{2,1} = \mu_2 \circ \pi_1$, where $\pi_1$ is the projection on $M_{\alpha_1}$.

We introduce the notion of multiplicity-free space for the reduced space. Indeed, we say that the $G_1$-action on $M_{\alpha_2}$ is multiplicity-free if the ring of $G_1$-invariant functions of $M_{\alpha_2}$, [1], is a commutative Poisson algebra.

We may also define the reduced space with respect the $G_1$-action on $M_{\alpha_2}$ to be

$$(M_{\alpha_2})_{\alpha_1} = \mu_{1,2}^{-1}(G_1\alpha_1)/G_1.$$

The same definition holds for the $G_2$-action on $M_{\alpha_1}$.

Now, we shall give a criterion for a $G$-action to be a multiplicity-free action. We begin with the following lemma.

**Lemma 4.1.** *Let $(M,\omega)$ be a symplectic manifold and let $G = G_1 \times G_2$ be a connected Lie group which acts in a Hamiltonian fashion on $M$. Let $\alpha_1 \in \mathfrak{g}_1^*$. Then the $G_2$-action on $M_{\alpha_1}$ is multiplicity-free if and only if for every $\alpha_2 \in \mathfrak{g}_2^*$ the reduced space $(M_{\alpha_1})_{\alpha_2}$ are points*

*Proof.* In the sequel we always refer to [1] and [2].

Let $\mu_2 : M \longrightarrow \mathfrak{g}_1^*$ be the moment map of the $G_1$-action and let $\mu_{2,1} : M_{\alpha_1} \longrightarrow \mathfrak{g}_2^*$ be the corresponding moment map of the $G_2$-action on the reduced space. We recall that the smooth function on the reduced spaces are defined by

$$C^\infty(M_{\alpha_1}) = C^\infty(M)^{G_1}|_{\mu_1^{-1}(G_1\alpha_1)}$$

and the reduced space is a locally finite union of symplectic manifolds (symplectic pieces) which are the following manifolds.

Let $H$ be a subgroup of $G$. The set $M^{(H)}$ of orbit of type $H$, i.e. the set of points which orbits are isomorphic to $G/H$, is a submanifold of $M$ [20]. The set $(\mu_1^{-1}(G_1\alpha_1) \cap M^{(H)})$ is a submanifold of constant rank and the quotient

$$(M_{\alpha_1})^{(H)} = (\mu_1^{-1}(G_1\alpha_1) \cap M^{(H)})/G_1,$$

is a symplectic manifold which inclusion $(M_{\alpha_1})^{(H)} \hookrightarrow M_{\alpha_1}$ is a Poisson map [2].

The $G_2$-action preserves $(M_{\alpha_1})^{(H)}$, and the following topological space

$$((M_{\alpha_1})^{(H)} \cap \mu_{2,1}^{-1}(G_2\alpha_2))/G_2 = \cup_{i\in I} S_i$$

is a stratified symplectic manifold which restrictions map

$$r^H_{\alpha_2} : C^\infty((M_{\alpha_1})^{(H)} \cap \mu_{2,1}^{-1}(G_1\alpha_1))^{G_2} \longrightarrow C^\infty(S_i)$$

are Poisson and surjective. Therefore, if $C^\infty(M_{\alpha_1})^{G_2}$ is abelian, the algebra $C^\infty(S_i),\ i \in I$ must be abelian, and $S_i$ must be discrete and therefore a points.

Vice-versa, if all reduced spaces are points then $r^H_{\alpha_2}(\{f,g\}) = 0$ for all $\alpha_2 \in g_2^*$, and every $H$ subgroup of $G_1$, so that $\{f,g\} = 0$. □

**Theorem 4.2.** *Let $(M,\omega)$ be a symplectic manifold with a proper and Hamiltonian action of a connected Lie group $G = G_1 \times G_2$. Assume also that $G_i,\ i = 1,2$ are closed connected Lie group and the coadjoint orbits of $G, G_1$ and $G_2$ are locally closed. Hence $M$ is a $G$ multiplicity-free space if and only if for every $\alpha = \alpha_1 + \alpha_2 \in \mathfrak{g}^*$ $M_{\alpha_1}$ is a $G_1$ multiplicity-free space if and only if $M_{\alpha_2}$ is a $G_2$ multiplicity-free space.*

*Proof.* It follows immediately from the above result. Indeed, it is easy to check that $M_\alpha = \mu^{-1}(G\alpha)/G$ is homemorphic to $(M_{\alpha_1})_{\alpha_2}$ or equivalently is homemorphic to $(M_{\alpha_2})_{\alpha_1}$; the homeomorphism is given by the natural application

$$(M_{\alpha_1})_{\alpha_2} \longrightarrow M_\alpha, \quad [[x]] \longrightarrow [x]$$

which preserves the symplectic pieces, concluding the prove. □

Theorem 4.2 is not difficult to prove. However, from Theorem 4.2, we may deduce some interesting facts.

**Proposition 4.3.** *Let $N$ be closed $G-$invariant symplectic submanifold of $M$. If $M$ is a $G$ multiplicity-free space then so is $N$.*

*Proof.* $G$ acts on $N$ in a Hamiltonian fashion with moment map $\overline{\mu} : N \longrightarrow \mathfrak{g}^*$, $\overline{\mu}(x) = \mu(x)$, which is the restriction of $\mu$ on $N$. In particular, for every $\alpha \in \mathfrak{g}^*$ the reduced space $N_\alpha \subset M_\alpha$, which implies that the topological space $N_\alpha$ are points. □

**Corollary 4.4.** *If $G$ is a compact Lie group acting multiplicity-free on $M$ then*

$$M^G := \{x \in M : Gx = x\},$$

*must be a finite set.*

Another interesting application of Theorem 4.2 is the following result.

**Proposition 4.5.** *Let $(M,\omega)$ be a symplectic manifold with a Hamiltonian circle action. Let $K$ be a connected Lie group which acts properly and in a Hamiltonian fashion on $M$. Assume also that the $K$-action commutes with the circle action. If $M$ is a $K$ multiplicity-free space then so is the $K$-action induced on any symplectic cut.*

*Proof.* We briefly describe the symplectic cut, see [17] and [7] for more details.

Let $(M,\omega)$ be a symplectic manifold with a Hamiltonian action of a one-dimensional torus $\mathrm{T}^1$ with moment map $\phi : M \longrightarrow \mathbb{R}$. We consider the symplectic manifold $N = M \times \mathbb{C}$, equipped with the symplectic form $\omega - \frac{i}{2} dz \wedge d\overline{z}$. $\mathrm{T}^1$ acts on $N$ with its product action and

this action is a Hamiltonian action with moment map $\psi(p,z)=\phi(p)+|z|^2$. The reduced space $M^\lambda=\psi^{-1}(\lambda)/S^1$, $\lambda\in\mathbb{R}$ is the symplectic cut.

The $K-$action on $M\times\mathbb{C}$ is given by $k(m,z)=(km,z)$. Since the $K-$action commutes with the $\mathrm{T}^1$-action, it induces a Hamiltonian action on the symplectic cut with moment map $\overline{\mu}([x,z])=\mu(x)$ where $\mu$ is the moment map of the $K-$action on $M$. Note that $K\times\mathrm{T}^1$-action is multiplicity-free on $M\times\mathbb{C}$ if and only if the $K$-action is on $M$. Therefore, if $K$ acts multiplicity-free on $M$, from Theorem 4.2, $K$ acts multiplicity-free on the symplectic cut. □

Let $H$ be a compact subgroup of $G$ and let $N(H)$ be its normalizer in $G$. It is well-known that the Lie group $L=N(H)/H$ acts freely and properly on the submanifold $M^H:=\{x\in M:G_x=H\}$ [20]. Moreover, since $T_xM^H=(T_xM)^H$, $M^H$ is symplectic.

In [2] it was proved that $L$ acts in a Hamiltonian fashion on $M^H$, the dual of the Lie algebra of $L$ is naturally isomorphic to the subspace $(\mathfrak{h}^o)^H$ of the $H$-fixed vectors in the annihilator of $\mathfrak{h}$=Lie($H$) in $\mathfrak{g}^*$. Furthermore, given $\alpha=\mu(x)$, where $x\in M^H$, then

$$G\mu^{-1}(\alpha)\cap M^H/G\cong(M^H)_{\alpha_o},$$

$\cong$ means symplectically diffeomorphic, where $\alpha_o=\pi(\alpha)$ and $\pi:(\mathfrak{h}^o)^H\longrightarrow l^*$ be the natural projection. This proves that if $M$ is a $G$ multiplicity-free space then, from Theorem 4.2, $M^H$ is a $L$ multiplicity-free space. Hence, we have the following result.

**Proposition 4.6.** *Let $H$ be a compact subgroup of $G$. If $G$ acts coisotropically on $M$ then $L$ acts coisotropically on $M^H$.*

Next, we claim the reduction principle for a multiplicity-free action.

**Proposition 4.7.** *(**Reduction principle**) Let $G$ be a connected Lie group acting properly on a connected symplectic manifold $M$. Let $H$ be a principal isotropy for the $G$-action. Then $G$ acts coisotropically on $M$ if and only if $L=N(H)/H$ acts coisotropically on $M^H$.*

*Proof.* Since the $G$-action is proper and preserves $\omega$, there exists a $G$-invariant almost complex structure $J$, i.e. $J:TM\longrightarrow TM$ be such that $J^2=-Id$, adapted to $\omega$, i.e. $\omega(J\cdot,J\cdot)=\omega(\cdot,\cdot)$ and $g=\omega(\cdot,J\cdot)$ is a Riemannian metric, see [2].

Now let $H$ be a principal isotropy and let $L=N(H)/H$. It is well-known that

$$M^H\cong N(H)/H\times(T_xGx)^{\perp_g},$$

see [20], which implies $T_xLx=(T_xGx)^H$.

Since $(T_xGx)^{\perp_\omega}=J((T_xGx)^{\perp_g})$ and $(T_xGx)^{\perp_g}\subset(T_xM)^H$, recall that $H$ acts trivially on the slice due the fact that $Gx$ is a principal orbit, we get that

$$J((T_xGx)^{\perp_g})\subset T_xGx\Leftrightarrow J((T_xGx)^{\perp_g})\subset(T_xGx)^H.$$

Therefore, recall that $(T_xLx)^{\perp_{g|_{T_xM^H}}}=(T_xGx)^{\perp_g}$ since $Gx$ is principal, we have that $Gx$ is coisotropic in $M$ if and only if $Lx$ is in $M^H$. □

We conclude this section giving the Equivariant mapping lemma, see [13], in a symplectic context.

**Proposition 4.8.** *Let $(M,\omega)$ and $(N,\omega_o)$ be connected symplectic manifolds and $G$ be a connected Lie group acting on both manifolds properly, and in a Hamiltonian fashion. Let*

*$\phi : M \longrightarrow N$ be a smooth surjective $G-$equivariant map with* $\mathrm{rank}\phi = \dim N$. *Assume that for every $p \in M$, $Kerd\phi_p$ is a symplectic subspace and*

$$d\phi_p : ((Ker\phi)^{\perp_\omega}, \omega) \longrightarrow (T_pN, \omega_o)$$

*is a symplectomorphism. If $M$ is a $G$ multiplicity-free space then so is $N$.*

*Proof.* Let $f \in C^\infty(N)^G$. The function $\tilde{f} = f \circ \phi$ is a $G$-invariant function of $M$. Take the vector field $X_f$ such that $df = i_{X_f}\omega_o$. By assumption the vector field $\tilde{X} \in (Ker\phi)^{\perp_\sigma}$ such that $d\phi(\tilde{X}) = X_f$ is the symplectic gradient of $\tilde{f}$. Hence, given $f, g \in C^\infty(N)^G$ there exist $\tilde{f}$, $\tilde{g} \in C^\infty(M)^G$ such that $\{f, g\}(\phi(x)) = \{\tilde{f}, \tilde{g}\}(x)$ which conclude our proof. □

## 5. multiplicity-free spaces in Kähler geometry

In the sequel we shall assume that $M$ is a compact Kähler manifold and $G$ is a closed subgroup of its full isometry group acting in a Hamiltonian fashion on $M$. Note that this action is automatically holomorphic by a Theorem of Konstant (see [16] p.242).

In [22] it was introduced the *homogeneity rank* of $(G, M)$ as the following integer

$$\mathrm{homrk}(G, M) = \mathrm{rk}(G) - \mathrm{rk}(G_{\mathrm{princ}}) - \mathrm{cohom}(G, M),$$

where $G_{\mathrm{princ}}$ is the principal isotropy subgroup of the action, the integer $\mathrm{cohom}(G, M)$ is the codimension of the principal orbit and, for a compact Lie group $H$, $\mathrm{rk}(H)$ denotes the rank, namely the dimension of the maximal torus.

In [13] it was proved that a group $G$ acts multiplicity-free on $M$ if and only if the homogeneity rank vanishes.

Our purpose is to make a connection between homogeneity rank of $(G, M)$ and homogeneity rank of $(G, M_\lambda)$, where $M_\lambda$ is the reduced space obtained from a torus action.

Let $K$ be a semisimple compact Lie subgroup of $G$ and let $\mathrm{T^k}$ be a $k$-dimensional connected torus which centralizes $K$ in $G$, i.e. $\mathrm{T^k} \subset \mathrm{C_G(K)}$. In the sequel we denote by

$$\phi : M \longrightarrow \mathfrak{k} \oplus \mathfrak{t}_k,$$

where $\mathfrak{t}_k = \mathrm{Lie}(\mathrm{T^k})$, the moment map of the $\mathrm{T^k} \cdot \mathrm{K}$-action on $M$ and with $\mu$, respectively with $\psi$, the moment map of the $K$-action on $M$, respectively a moment map of the $\mathrm{T^k}$-action on $M$.

Let $\lambda \in \mathfrak{t}_k$ be such that $\mathrm{T^k}$ acts freely on $\psi^{-1}(\lambda)$. The symplectic reduction

$$(M_\lambda = \psi^{-1}(\lambda)/\mathrm{T^k}, \omega_\lambda),$$

is a symplectic manifold and $\omega_\lambda$ satisfies

$$\pi^*(\omega_\lambda) = i^*(\omega),$$

where $\pi$ is the natural projection $\psi^{-1}(\lambda) \xrightarrow{\pi} M_\lambda$ and $i$ is the inclusion $M^\lambda \hookrightarrow M$, [8], [18]. Since $K$ commutes with $\mathrm{T^k}$, $K$ acts on $M_\lambda$ in a Hamiltonian fashion with moment map

$$\overline{\mu} : M_\lambda \longrightarrow \mathfrak{k},\ \overline{\mu}([x]) = \mu(x).$$

Indeed, It is easy to see that $\overline{\mu}$ is $K-$equivariant. Hence the problem is then restricted to verify that for every $Z \in \mathfrak{k}$ we have $d\overline{\psi}^Z = i_{\tilde{Z}^\#}\omega_\lambda$, where $\tilde{Z}^\#$ is the vector field on $M_\lambda$ generated by the one parameter subgroup $\exp(tZ)$.

Let $X \in T_{[x]}M_\lambda$ and let $\tilde{X} \in T_x\mu^{-1}(\lambda)$ such that $\pi_*(\tilde{X}) = X$. Since $\pi_*(Z^\#) = \tilde{Z}^\#$, where $Z^\#$ is the Killing field induced from $Z$ in $M$, it follows

$$d\overline{\psi}^Z(X) = d\psi^Z(\tilde{X}) = i_{Z^\#}\omega(\tilde{X}) = \pi^*\omega_\lambda(Z^\#, \tilde{X}) = i_{\tilde{Z}^\#}\omega_\lambda(X),$$

thus $K$ acts in a Hamiltonian fashion on $M^\lambda$.

Let $[p] \in M_\lambda$. It is easy to see that $k[p] = [p]$ if and only if there exists $r(k) \in \mathrm{T^k}$ such that $kp = r(k)p$, which is unique since $\mathrm{T^k}$ acts freely on $\psi^{-1}(\lambda)$. This means that the following application

$$K_{[p]} \xrightarrow{F} (\mathrm{T^k} \cdot \mathrm{K})_{\mathrm{p}},\ \mathrm{F(k) = kr(k)^{-1}}, \tag{3}$$

is a covering map, due the fact that $K$ is semisimple. Hence

$$\dim K[p] = \dim(\mathrm{T^k} \cdot \mathrm{K})\mathrm{p} - \dim \mathrm{T^k}. \tag{4}$$

Since $M$ is a compact manifold, we may extend the $\mathrm{T^k}$-action to a holomorphic action of $(\mathbb{C}^*)^n$ which commutes with the $K-$action. This can be easily deduced from the following easy fact: let $X, Y$ be holomorphic fields. If $[X, Y] = 0$ then $[X, J(Y)] = 0$, since $[X, J(Y)] = J([X, Y]) = 0$, due the fact that $M$ is Kähler. In particular the infinitesimal generatores of the $K-$action commute with ones of the $(\mathbb{C}^*)^n-$action, proving that the two action commute as well.

The set $(\mathbb{C}^*)^n \cdot \psi^{-1}(\lambda)$ is an open subset. Indeed, for every $p \in \psi^{-1}(\lambda)$, denoting with $\mathfrak{z}_p$ the vector subspace of $T_pM$ spanned by the infinitesimal generator of the $\mathrm{T^k}$-action on $M$, we have $T_p\psi^{-1}(\lambda) \oplus J(\mathfrak{z}_p) = T_pM$, since $\lambda$ is a regular value, which implies our affirmation. In particular the open subset $(\mathbb{C}^*)^n \cdot \psi^{-1}(\lambda)$ contains regular points. Hence there exists an element

$$q = \rho_1 \cdots \rho_n \exp(i\theta_1) \cdots \exp(i\theta_n)p = \rho \exp(i\Theta)p \in (\mathbb{C}^*)^n \cdot \psi^{-1}(\lambda),$$

such that the orbit $(\mathrm{T^k} \cdot \mathrm{K})\mathrm{q}$ is a principal orbit. Since $K$ commutes with $(\mathbb{C}^*)^n$, we get that $(\mathrm{T^k} \cdot \mathrm{K})_{\mathrm{p}} = (\mathrm{T^k} \cdot \mathrm{K})_{\mathrm{q}}$ which means that $p$, which lies in $\psi^{-1}(\lambda)$, is a regular point. Therefore, from (3) we deduce that $K[p]$ is a principal orbit and from (4), we get

$$\mathrm{homrk}(K, M_\lambda) = \mathrm{homrk}(\mathrm{T^k} \cdot \mathrm{K}, \mathrm{M}), \tag{5}$$

which proves the following result.

**Proposition 5.1.** *Let $G$ be a connected compact Lie group acting isometrically and in a Hamiltonian fashion on a compact Kähler manifold $M$. Let $K$ be a compact semisimple Lie group of $G$ which centralizer in $G$ contains a $k$-dimensional connected torus $\mathrm{T^k}$. Let $\lambda \in \mathfrak{t}_k$ be such that $\mathrm{T^k}$ acts freely on $\psi^{-1}(\lambda)$, where $\psi$ is a moment map of $\mathrm{T^k}$-action on $M$. Then the $(\mathrm{T^k} \cdot \mathrm{K})$-action is coisotropic on $M$ if and only if the $K$-action is on $M_\lambda = \psi^{-1}(\lambda)/\mathrm{T^k}$.*

If we consider a one-dimensional torus $\mathrm{T}^1$ we may investigate the $K$-action on the Kähler cut $M^\lambda$ obtained from the $\mathrm{T}^1$-action. Here we only assume that the $K$-action commutes with the $\mathrm{T}^1$-action. It is easy to check that $K_{[v,z]} = K_v$ when $z \neq 0$. Since $\{[(v, z] \in M^\lambda :\ z \neq 0\}$ is an open subset, one may deduce that

$$\mathrm{homrk}(K, M) = \mathrm{homrk}(K, M^\lambda). \tag{6}$$

Hence $K$ acts coisotropically on $M$ if and only if $K$ acts on $M^\lambda$ which proves the following result

**Proposition 5.2.** *Let $G$ be a connected compact Lie group acting isometrically and in a Hamiltonian fashion on a compact Kähler manifold $M$. Let $K$ be a compact Lie group of $G$ whose centralizer in $G$ contains a one-dimensional torus $\mathrm{T}^1$. Let $\lambda \in \mathfrak{t}_1$ be such that $\mathrm{T}^1$ acts freely on $\psi^{-1}(\lambda)$, where $\psi$ is a moment map of $\mathrm{T}^1$-action on $M$. Then $K$-action is coisotropic on $M$ if and only if the $K$-action on the Kähler cut $M_\lambda$ is.*

**Example 5.3.** Let $\omega = \sqrt{-1}\sum_{i=1}^{n+1} dz_i \wedge d\overline{z}_i$ be a Kähler form on $\mathbb{C}^{n+1}$. Consider the following $S^1-$action on $(\mathbb{C}^{n+1}, \omega)$ :

$$t \in S^1 \mapsto \theta_t = \text{multiplication by } e^{it}.$$

The action is Hamiltonian with moment map $\mu(z) = |z|^2 +$ constant. If we choose the constant to be $-1$, then $\mu^{-1}(0) = S^{2n+1}$ is the unit sphere on which $S^1$ acts freely and the Kähler reduction $\mu^{-1}(0)/S^1$ is just $\mathbb{P}^n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$. Therefore, by Proposition 5.1, a compact connected Lie subgroup $K$ of $\mathrm{SU}(n+1)$ acts multiplicity-free on $\mathbb{P}^n(\mathbb{C})$ if and only if $S^1 \cdot K$ acts multiplicity-free on $\mathbb{C}^n$. Kac [14] and Benson and Ratclif [3] have given the complete classification of linear coisotropic actions, from which one has the full classification of coisotropic actions on $\mathbb{P}^n(\mathbb{C})$.

If we consider the cut of $\mathbb{C}^{n+1}$ at $\lambda > 0$, with respect the above $\mathrm{S}^1$-action, we obtain, see [7], $\mathbb{P}^{n+1}(\mathbb{C})$, with $\lambda$ times the Fubini-Study metric. Hence $G \subset \mathrm{SU}(n+1)$ acts coisotropically on $\mathbb{P}^{n+1}(\mathbb{C})$ if and only if it acts coisotropically on $\mathbb{C}^{n+1}$.

## 6. Multiplicity-free actions on compact non Hermitian symmetric spaces

Let $\mathrm{T}^1$ acting on $\mathbb{P}^n(\mathbb{C})$, as

$$(t, [z_o, \ldots, z_n]) \longrightarrow [z_o, \cdots, tz_n].$$

This is a Hamiltonian action with moment map

$$\phi([z_o, \ldots, z_n]) = \frac{1}{2} \frac{\parallel z_n \parallel^2}{\parallel z_0 \parallel^2 + \cdots + \parallel z_n \parallel^2} \ .$$

Note that $\phi([0, \ldots, 1])$ is the maximum value of $\phi$ and $\phi^{-1}(\frac{1}{2}) = [0, \ldots, 1]$. Hence (see [7] page 5) if $\lambda = \frac{1}{2} - \epsilon$, $\epsilon \cong 0$, then the Kähler cut $\mathbb{P}^n(\mathbb{C})^\lambda$ is the blow up of $\mathbb{P}^n(\mathbb{C})$ at $[0, \ldots, 1]$.

Let $\mathrm{T}^\mathrm{n}$ be a torus acting on $\mathbb{P}^n(\mathbb{C})$ as follows

$$(t_1, \ldots, t_n)([z_o, \ldots, z_n]) = (t_1 z_o, t_2 z_1, \ldots, t_n z_{n-1}, z_n]).$$

This action is Hamiltonian and the principal orbits are Lagrangian; therefore $\mathrm{T}^\mathrm{n}$ acts coisotropically on $\mathbb{P}^n(\mathbb{C})$. Since $\mathrm{T}^\mathrm{n}$-action commutes with the above $\mathrm{T}^1$-action, from Proposition 5.2, we get $\mathrm{T}^\mathrm{n}$ acts coisotropically on the blow-up at one point of $\mathbb{P}^n(\mathbb{C})$.

We may generalize the above procedure as follows.

Let $G$ be a connected compact Lie group acting coistropically on a compact Kähler manifold. It is well-known that, see [12], $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, and if we denote by $G_{\mathrm{ss}}$ the semisimple connected compact Lie group whose Lie algebra is $[\mathfrak{g}, \mathfrak{g}]$, then

$$G = Z(G) \cdot G_{\mathrm{ss}}.$$

From Proposition 5.1, if $G_{ss}$ acts coisotropically on $M$, then so is the $G_{ss}$-action induced on $M^\lambda$, the reduced space obtained from $\mathrm{T}^\mathrm{k} \subset \mathrm{Z(G)}$.

Let $\mathrm{T}^1$ be a one-dimensional torus which lies on $Z(G)$. If $K \subset G$ is a compact Lie group acting coisotropically on $M$ then from Proposition 5.2 $K$ acts coisotropically on the Kähler cut, obtained from the $\mathrm{T}^1$-action on $M$. In particular, see [7], if $\lambda_o$ is a maximum for the

moment map of the $T^1$-action then $M^\lambda$, $\lambda = \lambda_o - \epsilon$, $\epsilon \cong 0$, is the blow-up of $M$ along the complex submanifold $\psi^{-1}(\lambda_o)$, where $\psi$ is the corresponding moment map for the $T^1$-action on $M$.

In [5],[6], [21], the complete classification of coisotropic actions on irreducible Hermitian symmetric spaces of compact type is given. Therefore, it is easy to construct examples using the above strategy. For example, the $\mathrm{SU}(n)$-action on $\mathrm{SO}(2n)/\mathrm{U}(n)$ induces a coisotropic action on Kähler cut given by $Z(\mathrm{U}(n))$. More generally, if $M = L/P$ is an irreducible Hermitian symmetric space of compact type, then $Z(P)$ is a one-dimensional torus. Since the $P$-action on $M$ is coisotropic, see [5], [6], [21], $P$ acts coisotropically on the Kähler cut with respect the $Z(P)$-action on $M$.

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